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Fermionic Path Integrals and Analytic Solutions for Two-Dimensional Ising Models †

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Abstract

The notion of the integral over anticommuting Grassmann variables (nonquantum fermionic fields) seems to be the most powerful tool in order to extract the exact solutions for the 2D Ising models on simple and more complicated lattices, which is the subject of a discussion in this report.

1. Introduction. The fermionic structures in the two-dimensional (2D) Ising model [1] were first recognized within the transfer-matrix and combinatorial approaches [2]. It was realized later on that the notion of the integral over anticommuting Grassmann variables due to Berezin [3] is a powerful tool to study the 2D Ising model [3-8]. A simple method of fermionic analysis for the 2D Ising models disposed on simple and more complicated lattices has been developed in [7-8]. The approach is based on the integration over the anticommuting Grassmann variables (nonquantum fermionic fields) and the mirror-ordering factorization principle for the 2DIM density matrix. The method is straightforward, the traditional transfer-matrix or combinatorial considerations are not used. Schematically, we have:

$$Q = \underset{(\sigma)}{\mathrm{Sp}}\; Q\,(\sigma) \to \underset{(\sigma\,|\,a)}{\mathrm{Sp}}\; Q\,(\sigma\,|\,a) \to \underset{(a)}{\mathrm{Sp}}\; Q\,(a) \,=\, Q\,. \tag{1}$$

We start here with the original Ising spin partition function, Q, and introduce, in a special way, a set of new purely anticommuting Grassmann variables (a) thus passing to a mixed $(\sigma\,|\,a)$ representation. Eliminating spin variables (σ) in this mixed representation, we obtain a purely fermionic expression for the same partition function Q. The final expression for Q appears as a fermionic Gaussian integral. In essence, this means the exact solution of the problem. The partition function is expressible here as a fermionic Gaussian integral even for the most general inhomogeneous distribution of coupling parameters [7]. In particular, for the standard rectangular lattice this gives a few line derivation of the Onsager result [7]. The 2D Ising lattices with more complicated local structures have been analyzed by the factorization method as well [8,9]. The Gaussian fermionic representations have recently been constructed also for the inhomogeneous 2D dimer models [10]. In this

scheme, we introduce Grassmann variables in order to decouple the bond Boltzmann weights into separable factors called the Grassmann factors (GFs). We then combine GFs with the same spin variables into separable groups and sum over spins in each group independently thus passing to a purely fermionic representation for Q. The key point of all the construction is the mirror-ordering procedure for the global products of GFs which enables one to perform the actual summing over spin degrees of freedom [7,8]. In what follows we comment shortly on some aspects of the method.

2. Grassmann variables. Grassmann variables are the purely anticommuting fermionic numbers. Given a set of Grassmann variables $a_1, a_2, ..., a_N$, we have $a_i a_j + a_j a_i = 0$, $a_j^2 = 0$. Berezin's rules of integration for one variable are [3]:

$$\int da_j \cdot a_j = 1 \,, \qquad \int da_j \cdot 1 = 0 \,. \tag{2}$$

In the multidimensional integral, the differentials $da_1, da_2, ..., da_N$ are again anticommuting with each other and with the variables. The basic relations of the grassmannian analysis concern the Gaussian fermionic integrals [3]. The Gaussian integral of the first kind is related to the determinant:

$$\int \prod_{j=1}^{N} da_j^* da_j \, \exp\left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j^* \right) \; = \; \det \hat{A} \,, \tag{3}$$

where $\{a_j, a_j^*\}$ is a set of completely anticommuting Grassmann variables, the matrix in the exponential is arbitrary. The fermionic exponential here is assumed in the sense of its series expansion, the series terminates at some stage due to the property $a_j^2 = 0$. The origin of the determinant in (3) is due to the known interrelation between the fermionic algebra and determinant combinatorics. By convention, the variables a_j and a_j^* can be considered as complex conjugated fermionic fields, otherwise these are simply independent variables. The Gaussian integral of the second kind, for real fermionic fields, is related to the Pfaffian:

$$\int da_N \ldots da_2 da_1 \, \exp\left(\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j \right) \; = \; \mathrm{Pfaff}\, \hat{A} \,, \qquad A_{ij} = -A_{ji} \,, \tag{4}$$

where the matrix is assumed to be skew-symmetric. The Pfaffian is some combinatorial polynomial in elements A_{ij} known in mathematics for a long time. Pfaffian combinatorics is in fact identical with that of the fermionic version of Wick's theorem. For any skew-symmetric matrix ($A_{ij} = -A_{ji}$) we have: $\det \hat{A} = (\, \mathrm{Pfaff}\, \hat{A}\,)^2$.

3. The 2D Ising model on rectangular lattice. To illustrate (1), let us consider the 2D Ising model on a rectangular lattice net with the inhomogeneous distribution of coupling parameters [7]. The hamiltonian is given as follows:

$$-\beta\, H\,(\sigma) = \sum_{m=1}^{L} \sum_{n=1}^{L} [\, b^{(1)}_{m+1n} \sigma_{mn} \sigma_{m+1n} + b^{(2)}_{mn+1} \sigma_{mn} \sigma_{mn+1} \,] \,, \;\; \beta = 1/kT \,, \tag{5}$$

where $b^{(\alpha)}_{mn} = J^{(\alpha)}_{mn}/kT$, $J^{(\alpha)}_{mn}$ are the magnetic exchange energies, $\beta = 1/kT$ is the inverse temperature. The Ising spins $\sigma_{mn} = \pm 1$ are disposed at lattice sites mn, with

$m, n = 1, ..., L$, $N = L^2 \to \infty$. For finite N, we assume free-boundary conditions: $\sigma_{M+1n} = \sigma_{mN+1} = 0$. Noting that for typical bond weight $\mathrm{e}^{b\sigma\sigma'} = \cosh b + \sinh b\, \sigma\sigma'$, which readily follows from $(\sigma\sigma')^2 = 1$, we come to the reduced partition function:

$$Q = \mathop{\mathrm{Sp}}_{(\sigma)} \, \{ \prod_{mn} (\, 1 + t^{(1)}_{m+1n}\sigma_{mn}\sigma_{m+1n}) \, (1 + t^{(2)}_{mn+1}\sigma_{mn}\sigma_{mn+1}) \, \} \, , \tag{6}$$

where $t^{(\alpha)}_{mn} = \tanh\, b^{(\alpha)}_{mn}$, and $Sp_{(\sigma)} = 2^{-N}\Sigma_{(\sigma)}$ stands for a properly normalized spin averaging [7]. Following (1), we introduce a set of Grassmann variables $\{a_{mn},\, a^*_{mn},\, b_{mn},\, b^*_{mn}\}$, a pair per bond, and factorize the local bond weights as follows:

$$\begin{aligned} 1 + t^{(1)}_{m+1n}\sigma_{mn}\sigma_{m+1n} &= \int da^*_{mn} da_{mn}\, \mathrm{e}^{a_{mn}a^*_{mn}}\, (1 + a_{mn}\sigma_{mn})\, (1 + t^{(1)}_{m+1n}\, a^*_{mn}\sigma_{m+1n})\, , \\ 1 + t^{(2)}_{mn+1}\sigma_{mn}\sigma_{mn+1} &= \int db^*_{mn} db_{mn}\, \mathrm{e}^{b_{mn}b^*_{mn}}\, (1 + b_{mn}\sigma_{mn})\, (1 + t^{(2)}_{mn+1}\, b^*_{mn}\sigma_{mn+1})\, . \end{aligned} \tag{7}$$

These identities can be checked simply by using the rules (2), and noting that $\mathrm{e}^{aa^*} = 1 + aa^*$, since $(aa^*)^2 = 0$. Neglecting the sign of the Gaussian fermionic averaging, we see that the bond weights are presented now as $A_{mn}A^*_{m+1n}$, $B_{mn}B^*_{mn+1}$, where the Grassmann factors A_{mn}, A^*_{m+1n}, B_{mn}, B^*_{mn+1} are to be identified ¿from (7).

At the next stage, we group together, over the whole lattice, the four factors with the same Ising spin σ_{mn}. These four factors come by the factorization of the four different bonds (weights) attached to a given mn site. Performing the averaging over $\sigma_{mn} \pm 1$ in each group, we find [7]:

$$\begin{aligned} &\mathop{\mathrm{Sp}}_{(\sigma_{mn})} \{ A^*_{mn}B^*_{mn}A_{mn}B_{mn} \} = \tfrac{1}{2} \sum_{\sigma_{mn}=\pm 1} (1 + t^{(1)}_{mn}\, \sigma_{mn}a^*_{m-1n})\, (1 + t^{(2)}_{mn}\, \sigma_{mn}b^*_{mn-1}) \\ &\times (1 + \sigma_{mn}a_{mn})\, (1 + \sigma_{mn}b_{mn}) = 1 + t^{(1)}_{mn}t^{(2)}_{mn}\, a^*_{m-1n}b^*_{mn-1} + a_{mn}b_{mn} + \\ &+ (t^{(1)}_{mn}a^*_{m-1n} + t^{(2)}_{mn}b^*_{mn-1})\, (a_{mn} + b_{mn}) \, + \, t^{(1)}_{mn}t^{(2)}_{mn}\, a^*_{m-1n}b^*_{mn-1}a_{mn}b_{mn} = \\ &= \exp\, [a_{mn}b_{mn} + t^{(1)}_{mn}t^{(2)}_{mn}a^*_{m-1n}b^*_{mn-1} + (t^{(1)}_{mn}a^*_{m-1n} + t^{(2)}_{mn}b^*_{mn-1})\, (a_{mn} + b_{mn})]\, . \end{aligned} \tag{8}$$

Taking also into account the diagonal Gaussian terms arising under factorization (7), we obtain the partition function (6) in the form [7]:

$$\begin{aligned} Q \; = \; &\int \prod_{m=1}^{L} \prod_{n=1}^{L} db^*_{mn}db_{mn}da^*_{mn}da_{mn} \; \exp\, \{ \sum_{m=1}^{L} \sum_{n=1}^{L} [\, a_{mn}a^*_{mn} + b_{mn}b^*_{mn} + \\ &+ a_{mn}b_{mn} + t^{(1)}_{mn}t^{(2)}_{mn}a^*_{m-1n}b^*_{mn-1} + \, (t^{(1)}_{mn}a^*_{m-1n} + t^{(2)}_{mn}b^*_{mn-1})\, (a_{mn} + b_{mn})\,]\, \} \, , \end{aligned} \tag{9}$$

with the free-boundary condition for fermions: $a^*_{0n} = 0$, $b^*_{m0} = 0$. Now Q is given as a Gaussian fermionic integral, equivalently, the 2D Ising model is reformulated as a free fermion field theory on a lattice. For space restriction, we do not comment here on the mirror-ordering procedure which just makes it possible to put the four relevant factors nearby as in (8), see for details [7,8]. It is important that the method works for the inhomogeneous lattices, this might be of interest, in particular, for studies of random systems [6,11]. The integral (9) can in fact be simplified integrating out the a_{mn}, b_{mn} fields by means of the identity $\int db\, da\, \exp\,(ab + aL + L'b) = \exp\,(LL')$, where L, L' are some linear forms in Grassmann variables independent of a, b.

4. The 2D Ising models on complicated lattices. An important modification of the method has been introduced in [8], where we start with the factorization of the *cell* weights presented by three-spin polynomials. The spin polynomial interpretation arises if we multiply few local weights forming elementary cell in Q. This enables us to obtain the fermionic representation for Q with only two fermionic variables *per site*, which provides essential simplifications in the analysis [8]. For a set of 2D Ising models, including the standard rectangular, triangular and hexagonal lattices as the simplest cases, the partition function then reduces to the following expression [8]:

$$Q = \underset{(\sigma)}{\mathrm{Sp}} \, \Big\{ \prod_{mn} (\alpha_0 + \alpha_1 \, \sigma_{mn}\sigma_{m+1n} + \alpha_2 \, \sigma_{m+1n}\sigma_{m+1n+1} + \alpha_3 \, \sigma_{mn}\sigma_{m+1\, n+1}) \Big\} \, , \tag{10}$$

where α_0, α_1, α_2, α_3 are numerical parameters specific for each lattice (we now assume the homogeneous case). The three-spin polynomial in (10) is the effective Boltzmann weight of the elementary cell. The factorization method then yields the following Gaussian fermionic integral for the partition function [8]:

$$\begin{aligned} Q \, = \int \prod_{mn} dc^*_{mn} dc_{mn} \exp \Big\{ \sum_{mn} \big[\, \alpha_0 \, c_{mn} c^*_{mn} - \alpha_1 \, c_{mn} c^*_{m-1n} - \alpha_2 \, c_{mn} c^*_{mn-1} - \\ - \alpha_3 \, c_{mn} c^*_{m-1n-1} - \alpha_1 \, c_{mn} c_{m-1n} - \, \alpha_2 \, c^*_{mn} c^*_{mn-1} \big] \Big\} \, , \end{aligned} \tag{11}$$

where c_{mn}, c^*_{mn} is a set of the purely anticommuting Grassmann variables (two per cell). The derivation of (11) is simple [8].

The explicit evaluation of the integral (11) can be performed by passing to the momentum space (Fourier substitution for fermions). Taking then the limit of infinite lattice, we find the free energy per cell as follows [8]:

$$\begin{aligned} & -\beta f_Q = \lim{}_{L\to\infty} \left(\tfrac{1}{L^2} \ln Q \, \right) \, = \\ & = \tfrac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \tfrac{dp}{2\pi} \tfrac{dq}{2\pi} \, \ln \big[\, (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2) - 2 \, (\alpha_0 \alpha_1 - \alpha_2 \alpha_3) \, \cos p - \\ & - 2 \, (\alpha_0 \alpha_2 - \alpha_1 \alpha_3) \, \cos q \, - 2 \, (\alpha_0 \alpha_3 - \alpha_1 \alpha_2) \, \cos \, (p+q) \, \big] \, . \end{aligned} \tag{12}$$

The symmetries of this solution, and the closely related question on the location of the critical point, have an interesting interpretation within the spin-polynomial approach, as is discussed in [8], also see [12]. By a suitable specification of the parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ from (12) we obtain, in particular, the solutions for the free energies for canonical rectangular, triangular, and hexagonal Ising lattices, as well as for some other lattices with more complicated local structures [8]. For application of these results to the analysis of the regularly diluted 2D Ising ferromagnets also see [9]. The fermionic integral for Q with the minimal number of the fermionic components (11) appears to be a suitable starting point to formulate the continuum-limit field theories for 2D Ising models near T_c, as is discussed in more detail in [12]. The resulting field theory is the massive two-component Majorana theory in the 2D euclidean space-time [12]. By doubling of fermions, we can pass as well to the 2D Dirac field theory of charged fermions [12].

5. Conclusions. We have discussed some aspects of a simple non-combinatorial approach to the analytic solutions for the 2D Ising models on simple and more

complicated lattices. For any planar 2D Ising model the partition function can be expressed as a fermionic Gaussian integral. The integral over the anticommuting Grassmann fields is a powerful tool to analyze the two-dimensional Ising models.

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